

MULTILINEAR DYADIC OPERATORS AND THEIR COMMUTATORS IN THE WEIGHTED SETTING

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ABSTRACT. In this article, we investigate the boundedness properties of the multilinear dyadic paraproduct operators in the weighted setting. We also obtain weighted estimates for the multilinear Haar multipliers and their commutators with dyadic BMO functions.

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1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

The purpose of this article is to investigate the boundedness properties of the multilinear dyadic operators (paraproducts and Haar multipliers) introduced in [9] and their commutators in the weighted setting as adopted in [10]. Mainly, we use the unweighted theory of multilinear dyadic operators from [9], explore some useful properties of those operators, and run the machinery used in [10] to obtain the corresponding weighted theory of the multilinear dyadic operators.

In [9], the paraproduct decomposition of the pointwise product of two functions was generalized to the product of $m \geq 2$ functions that served as the motivation for defining the following multilinear dyadic operators.

- $P^{\vec{\alpha}}(f_1, f_2, \dots, f_m) = \sum_{I \in \mathcal{D}} \left(\prod_{j=1}^m f_j(I, \alpha_j) \right) h_I^{\sigma(\vec{\alpha})}, \quad \vec{\alpha} \in \{0, 1\}^m \setminus \{(1, 1, \dots, 1)\}.$
- $\pi_b^{\vec{\alpha}}(f_1, f_2, \dots, f_m) = \sum_{I \in \mathcal{D}} \langle b, h_I \rangle \left(\prod_{j=1}^m f_j(I, \alpha_j) \right) h_I^{1+\sigma(\vec{\alpha})}, \quad \vec{\alpha} \in \{0, 1\}^m, b \in BMO^d.$
- $T_\epsilon^{\vec{\alpha}}(f_1, f_2, \dots, f_m) := \sum_{I \in \mathcal{D}} \epsilon_I \left(\prod_{j=1}^m f_j(I, \alpha_j) \right) h_I^{\sigma(\vec{\alpha})},$

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$$\vec{\alpha} \in \{0, 1\}^m \setminus \{(1, 1, \dots, 1)\}, \epsilon = \{\epsilon_I\}_{I \in \mathcal{D}} \text{ bounded.}$$

$$\bullet [b, T_\epsilon^{\vec{\alpha}}]_i(f_1, f_2, \dots, f_m)(x) := b(x)T_\epsilon^{\vec{\alpha}}(f_1, f_2, \dots, f_m)(x) - T_\epsilon^{\vec{\alpha}}(f_1, \dots, bf_i, \dots, f_m)(x),$$

$$1 \leq i \leq m, \vec{\alpha} \in \{0, 1\}^m \setminus \{(1, 1, \dots, 1)\}, \epsilon = \{\epsilon_I\}_{I \in \mathcal{D}} \text{ bounded and } b \in BMO^d.$$

In the above definitions, $\mathcal{D} := \{[m2^{-k}, (m+1)2^{-k}) : m, k \in \mathbb{Z}\}$ is the standard dyadic grid on \mathbb{R} and h_I 's are the Haar functions defined by $h_I = \frac{1}{|I|^{1/2}} (1_{I_+} - 1_{I_-})$, where I_- and I_+ are the left and right halves of I . With $\langle \cdot, \cdot \rangle$ denoting the standard inner product in $L^2(\mathbb{R})$, $f_i(I, 0) := \langle f_i, h_I \rangle$ and $f_i(I, 1) := \langle f_i, h_I^2 \rangle = \frac{1}{|I|} \int_I f_i$, the average of f_i over I . The Haar coefficient $\langle f_i, h_I \rangle$ is sometimes denoted by $\widehat{f_i}(I)$ and the average of f_i over I by $\langle f_i \rangle_I$. For $\vec{\alpha} \in \{0, 1\}^m$, $\sigma(\vec{\alpha})$ denotes the number of 0 components in $\vec{\alpha}$. For convenience, we will denote the set $\{0, 1\}^m \setminus \{(1, 1, \dots, 1)\}$ by U_m .

The following boundedness properties of the multilinear dyadic operators were proved in [9]:

- Let $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \{0, 1\}^m$ and $1 < p_1, p_2, \dots, p_m < \infty$ with $\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{p}$. Then

(a) For $\vec{\alpha} \neq (1, 1, \dots, 1)$,

$$\|P^{\vec{\alpha}}(f_1, f_2, \dots, f_m)\|_p \lesssim \prod_{j=1}^m \|f_j\|_{p_j}.$$

(b) For $\sigma(\vec{\alpha}) \leq 1$,

$$\|\pi_b^{\vec{\alpha}}(f_1, f_2, \dots, f_m)\|_p \lesssim \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{p_j},$$

if and only if $b \in BMO^d$.

For $\sigma(\vec{\alpha}) > 1$,

$$\|\pi_b^{\vec{\alpha}}(f_1, f_2, \dots, f_m)\|_p \leq C_b \prod_{j=1}^m \|f_j\|_{p_j},$$

if and only if $\sup_{I \in \mathcal{D}} \frac{|\langle b, h_I \rangle|}{\sqrt{|I|}} < \infty$.

(c) For $\vec{\alpha} \neq (1, 1, \dots, 1)$,

$$\|T_\epsilon^{\vec{\alpha}}(f_1, f_2, \dots, f_m)\|_p \lesssim \prod_{j=1}^m \|f_j\|_{p_j},$$

if and only if $\|\epsilon\|_\infty := \sup_{I \in \mathcal{D}} |\epsilon_I| < \infty$.

In each of the above cases, the operators have the corresponding weak-type boundedness from $L^{p_1} \times \cdots \times L^{p_m} \rightarrow L^{p,\infty}$ if $1 \leq p_1, p_2, \dots, p_m < \infty$.

- Let $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m) \in U_m$. If $b \in BMO^d \cap L^r$ for some $1 < r < \infty$ and $\|\epsilon\|_\infty := \sup_{I \in \mathcal{D}} |\epsilon_I| < \infty$, then each commutator $[b, T_\epsilon^{\vec{\alpha}}]_i$ is bounded from $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m} \rightarrow L^r$ for all $1 < p_1, p_2, \dots, p_m, p < \infty$ with

$$\sum_{j=1}^m \frac{1}{p_j} = \frac{1}{p},$$

with estimates of the form:

$$\|[b, T_\epsilon^{\vec{\alpha}}]_i(f_1, f_2, \dots, f_m)\|_p \lesssim \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{p_j}.$$

In the above results L^p stands for the Lebesgue space $L^p(\mathbb{R}) := \{f : \|f\|_p < \infty\}$ with $\|f\|_p = \|f\|_{L^p} := \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{1/p}$. The Weak L^p space, also denoted by $L^{p,\infty}$, is the space of all functions f such that

$$\|f\|_{L^{p,\infty}(\mathbb{R})} := \sup_{t>0} t |\{x \in \mathbb{R} : f(x) > t\}|^{1/p} < \infty.$$

Moreover, $\|b\|_{BMO^d} := \sup_{I \in \mathcal{D}} \frac{1}{|I|} \int_I |b(x) - \langle b \rangle_I| dx < \infty$, is the dyadic BMO norm of b .

For the theory of linear operators, we refer to [11] and [7].

In [10], the concept of *multilinear $A_{\vec{P}}$ condition* was introduced to study the boundedness properties of the multilinear Calderón-Zygmund operators and their commutators. The use of multi(sub)linear maximal function was key to obtain the weighted estimates for the optimal range. The multilinear $A_{\vec{P}}$ condition is as follows:

Let $\vec{P} = (p_1, \dots, p_m)$ and $\vec{w} = (w_1, \dots, w_m)$, where $1 \leq p_1, \dots, p_m < \infty$ with $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$, and w_1, \dots, w_m are non-negative measurable functions. We say that \vec{w} satisfies the multilinear $A_{\vec{P}}$ condition and we write $\vec{w} \in A_{\vec{P}}$ if

$$\sup_I \left(\frac{1}{|I|} \int_I \nu_{\vec{w}} \right)^{\frac{1}{p}} \prod_{j=1}^m \left(\frac{1}{|I|} \int_I w_j^{1-p'_j} \right)^{\frac{1}{p'_j}} < \infty,$$

where $\nu_{\vec{w}} := \prod_{j=1}^m w_j^{p/p_j}$, and $\left(\frac{1}{|I|} \int_I w_j^{1-p'_j} \right)^{\frac{1}{p'_j}}$ is understood as $\|w_j^{-1}\|_{L^\infty(I)}$ when $p_j = 1$.

We define the corresponding dyadic multilinear $A_{\vec{P}}$ class, denoted by $A_{\vec{P}}^d$, by restricting the above definition to the dyadic intervals I .

We now state the main results of this article, which are the dyadic analogues of the corresponding results for the multilinear Calderón-Zygmund operators obtained in [10].

Theorem: Let $b \in BMO^d$, and $\epsilon = (\epsilon_I)_{I \in \mathcal{D}}$ be bounded. Suppose $T \in \{P^{\vec{\alpha}}, T_\epsilon^{\vec{\alpha}}\}$ with $\vec{\alpha} \in U_m$, or $T = \pi_b^{\vec{\alpha}}$ with $\vec{\alpha} \in \{0, 1\}^m$. Let $\vec{w} = (w_1, \dots, w_m) \in A_{\vec{P}}^d$ for $\vec{P} = (p_1, \dots, p_m)$ with $\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p}$.

(a) If $1 < p_1, \dots, p_m < \infty$, then

$$\|T(f_1, \dots, f_m)\|_{L^p(\nu_{\vec{w}})} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

(b) If $1 \leq p_1, \dots, p_m < \infty$, then

$$\|T(f_1, \dots, f_m)\|_{L^{p,\infty}(\nu_{\vec{w}})} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

Theorem: Let $\vec{\alpha} \in U_m$ and $\epsilon = (\epsilon_I)_{I \in \mathcal{D}}$ be bounded. Suppose $b \in BMO^d$ and $\vec{w} = (w_1, \dots, w_m) \in A_{\vec{P}}^d$ for $\vec{P} = (p_1, \dots, p_m)$ with $\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p}$ and $1 < p_1, \dots, p_m < \infty$. Then there exists a constant C such that

$$\|[b, T_\epsilon^{\vec{\alpha}}]_i(f_1, \dots, f_m)\|_{L^p(\nu_{\vec{w}})} \leq C \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

In the above results $L^p(w)$ stands for the weighted Lebesgue space $L^p(\mathbb{R}, w) := \{f : \|f\|_{L^p(w)} < \infty\}$ with $\|f\|_{L^p(w)} := \left(\int_{\mathbb{R}} |f(x)|^p w(x) dx \right)^{1/p}$. Moreover, the weak space $L^{p,\infty}(w)$ is the space of all functions f such that

$$\|f\|_{L^{p,\infty}(w)} := \sup_{t>0} t w(\{x \in \mathbb{R} : f(x) > t\})^{1/p} < \infty.$$

We organize the article as follows:

In section 2, we present an overview of the basic terms and related facts we will be using in this article. These include the Haar system, various maximal operators, A_p and multilinear $A_{\vec{P}}$ classes, and the BMO space.

In section 3, we investigate boundedness properties of the multilinear dyadic operators in the weighted setting. Weighted estimates for the commutator of the multilinear Haar multiplier with a dyadic BMO function are explored in section 4.

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2. PRELIMINARIES

2.1. The Haar System. Let \mathcal{D} denote the standard dyadic grid on \mathbb{R} ,

$$\mathcal{D} = \{[m2^{-k}, (m+1)2^{-k}) : m, k \in \mathbb{Z}\}.$$

Associated to each dyadic interval I there is a Haar function h_I defined by

$$h_I(x) = \frac{1}{|I|^{1/2}} (1_{I_+} - 1_{I_-}),$$

where I_- and I_+ are the left and right halves of I .

The collection of all Haar functions $\{h_I : I \in \mathcal{D}\}$ is an orthonormal basis of $L^2(\mathbb{R})$, and an unconditional basis of L^p for $1 < p < \infty$. In fact, if a sequence $\epsilon = \{\epsilon_I\}_{I \in \mathcal{D}}$ is bounded, the operator T_ϵ defined by

$$T_\epsilon f(x) = \sum_{I \in \mathcal{D}} \epsilon_I \langle f, h_I \rangle h_I$$

is bounded in L^p for all $1 < p < \infty$. The converse also holds. The operator T_ϵ is called the Haar multiplier with symbol ϵ .

2.2. A_p classes. A weight w is a non-negative locally integrable function on \mathbb{R} such that $0 < w(x) < \infty$ for almost every x . Given a weight w and a measurable set $E \subseteq \mathbb{R}$, the w -measure of E is defined by

$$w(E) = \int_E w(x) dx.$$

We say that a weight w belongs to the class A_p for $1 < p < \infty$ if it satisfies the Muckenhoupt condition:

$$\sup_I \left(\frac{1}{|I|} \int_I w \right) \left(\frac{1}{|I|} \int_I w^{-\frac{1}{p-1}} \right)^{p-1} < \infty,$$

where the supremum is taken over all intervals. The expression on the left is called the A_p (Muckenhoupt) characteristic constant of w , and is denoted by $[w]_{A_p}$. Note that if p' is the conjugate index of p , i.e. $\frac{1}{p} + \frac{1}{p'} = 1$, then $1 - p' = -\frac{1}{p-1} = -\frac{p'}{p}$. So,

$$\begin{aligned} [w]_{A_p} &= \sup_I \left(\frac{1}{|I|} \int_I w \right) \left(\frac{1}{|I|} \int_I w^{1-p'} \right)^{1/p'} \\ &= \sup_I \left(\frac{1}{|I|} \int_I w \right) \left(\frac{1}{|I|} \int_I w^{-\frac{p'}{p}} \right)^{\frac{p}{p'}}. \end{aligned}$$

It can be shown that $\lim_{p \rightarrow 1} \left(\frac{1}{|I|} \int_I w^{-\frac{1}{p-1}} \right)^{p-1} = \|w^{-1}\|_{L^\infty(I)}$. This leads to the following definition of A_1 class:

A weight w is called an A_1 weight if

$$[w]_{A_1} := \sup_I \left(\frac{1}{|I|} \int_I w \right) \|w^{-1}\|_{L^\infty(I)} < \infty.$$

Thus $[w]_{A_1}$ is the infimum of all constants C such that for all intervals I ,

$$\frac{1}{|I|} \int_I w \leq Cw(x) \quad \text{for a.e. } x \in I.$$

The A_p classes are increasing with respect to p , i.e. for $1 \leq p_1 < p_2 < \infty$,

$$[w]_{A_{p_2}} \leq [w]_{A_{p_1}}.$$

It is natural to define the A_∞ class of weights by

$$A_\infty = \bigcup_{p>1} A_p,$$

with $[w]_{A_\infty} = \inf\{[w]_{A_p} : w \in A_p\}$.

For $1 \leq p < \infty$, the dyadic A_p^d classes are defined by the same inequalities restricted to the dyadic intervals. Moreover, $A_\infty^d = \bigcup_{p>1} A_p^d$.

2.3. Multilinear $A_{\vec{p}}$ condition. We recall the multilinear $A_{\vec{p}}$ condition introduced by Lerner et al. [10].

Let $\vec{P} = (p_1, \dots, p_m)$ and $\vec{w} = (w_1, \dots, w_m)$, where $1 \leq p_1, \dots, p_m < \infty$ and w_1, \dots, w_m are non-negative measurable functions. Let $\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p}$.

We say that \vec{w} satisfies the multilinear $A_{\vec{p}}$ condition and we write $\vec{w} \in A_{\vec{p}}$ if

$$\sup_I \left(\frac{1}{|I|} \int_I \nu_{\vec{w}} \right)^{\frac{1}{p}} \prod_{j=1}^m \left(\frac{1}{|I|} \int_I w_j^{1-p'_j} \right)^{\frac{1}{p'_j}} < \infty,$$

where $\nu_{\vec{w}} := \prod_{j=1}^m w_j^{p/p_j}$, and $\left(\frac{1}{|I|} \int_I w_j^{1-p'_j} \right)^{\frac{1}{p'_j}}$ is understood as $\|w_j^{-1}\|_{L^\infty(I)}$ when $p_j = 1$.

Using Hölder's inequality, it is easy to see that

$$\prod_{j=1}^m A_{p_j} \subset A_{\vec{p}}.$$

Moreover, if $\vec{w} \in A_{\vec{p}}$, $\nu_{\vec{w}} \in A_{mp}$. We will denote the *dyadic multilinear $A_{\vec{p}}$ class* by $A_{\vec{p}}^d$.

2.4. Maximal Operators. Given a function f , the maximal function Mf is defined by

$$Mf(x) := \sup_{I \ni x} \frac{1}{|I|} \int_I |f(t)| dt,$$

where the supremum is taken over all intervals I in \mathbb{R} that contain x .

For $\delta > 0$, the maximal operator M_δ is defined by

$$M_\delta f(x) := M(|f|^\delta)^{1/\delta}(x) = \left(\sup_{I \ni x} \frac{1}{|I|} \int_I |f(t)|^\delta dt \right)^{1/\delta}.$$

The sharp maximal function $M^\#$ is given by

$$M^\# f(x) := \sup_{I \ni x} \inf_c \frac{1}{|I|} \int_I |f(t) - c| dt.$$

In fact,

$$M^\# f(x) \approx \sup_{I \ni x} \frac{1}{|I|} \int_I |f(t) - \langle f \rangle_I| dt,$$

where $\langle f \rangle_I := \frac{1}{|I|} \int_I f(t) dt$ is the average of f over I .

Given $\vec{f} = (f_1, \dots, f_m)$, the maximal operators \mathcal{M} and \mathcal{M}_r with $r > 0$ are defined by

$$\mathcal{M}(\vec{f})(x) = \sup_{I \ni x} \prod_{i=1}^m \frac{1}{|I|} \int_I |f_i(y_i)| dy_i$$

and

$$\mathcal{M}_r(\vec{f})(x) = \sup_{I \ni x} \prod_{i=1}^m \left(\frac{1}{|I|} \int_I |f_i(y_i)|^r dy_i \right)^{1/r}.$$

We will be using dyadic versions of the above maximal operators which are defined by taking supremum over all dyadic intervals $I \ni x$, instead of all intervals $I \ni x$. For convenience, we will use the same notation to denote the dyadic counterparts.

We will use the following results regarding maximal functions. The dyadic analogs of these statements are also true.

- For any locally integrable function f , $|f(x)| \leq Mf(x)$ almost everywhere. This inequality is a consequence of Lebesgue differentiation theorem and can be found in any standard Fourier Analysis textbooks, see for example [2] or [4]. In fact, for any $\delta > 0$, if $f \in L_{loc}^\delta(\mathbb{R})$, then $|f(x)| \leq M_\delta f(x)$ almost everywhere.
- For $0 < \delta_1 < \delta_2 < \infty$, $M_{\delta_1} f(x) \leq M_{\delta_2} f(x)$. This simple inequality can be verified just by using Hölder's inequality.
- For $w \in A_p$ with $1 < p < \infty$ there exists a constant C such that

$$\|Mf\|_{L^p(w)} \leq C \|f\|_{L^p(w)}. \quad (\text{See [11], [2]})$$

- Fefferman-Stein's inequalities (see [3]): Let $w \in A_\infty$ and $0 < \delta, p < \infty$. Then there exists a constant C_1 such that

$$(2.1) \quad \|M_\delta f\|_{L^p(w)} \leq C_1 \|M_\delta^\# f\|_{L^p(w)}$$

for all functions f for which the left-hand side is finite.

Similarly, there exists a constant C_2 such that

$$(2.2) \quad \|M_\delta f\|_{L^{p,\infty}(w)} \leq C_2 \|M_\delta^\# f\|_{L^{p,\infty}(w)}$$

for all functions f for which the left-hand side is finite.

- Let $\vec{P} = (p_1, \dots, p_m)$ and $\vec{w} = (w_1, \dots, w_m)$, where $1 < p_1, \dots, p_m < \infty$ with $\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p}$, and w_1, \dots, w_m are weights. Then the inequality

$$(2.3) \quad \|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}$$

holds for every $\vec{f} = (f_1, \dots, f_m)$ if and only if $\vec{w} \in A_{\vec{P}}$. For $1 \leq p_1, \dots, p_m < \infty$, the same statement is true with the inequality

$$(2.4) \quad \|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{w}})} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

These estimates and the one below have been obtained in [10].

- If $\vec{w} = (w_1, \dots, w_m) \in A_{\vec{P}}$, for $\vec{P} = (p_1, \dots, p_m)$ with $1 < p_1, \dots, p_m < \infty$ and $\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p}$, then there exists an $r > 1$ such that $\vec{w} \in A_{\vec{P}/r}$, and that

$$(2.5) \quad \|\mathcal{M}_r(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

2.5. BMO Space. A locally integrable function b is said to be of bounded mean oscillation if

$$\|b\|_{BMO} := \sup_I \frac{1}{|I|} \int_I |b(x) - \langle b \rangle_I| dx < \infty,$$

where the supremum is taken over all intervals in \mathbb{R} . The space of all functions of bounded mean oscillation is denoted by BMO .

If we take the supremum over all dyadic intervals in \mathbb{R} , we get a larger space of dyadic BMO functions which we denote by BMO^d .

For $0 < r < \infty$, define

$$BMO_r = \{b \in L^r_{loc}(\mathbb{R}) : \|b\|_{BMO_r} < \infty\},$$

where, $\|b\|_{BMO_r} := \left(\sup_I \frac{1}{|I|} \int_I |b(x) - \langle b \rangle_I|^r dx \right)^{1/r}$.

For any $0 < r < \infty$, the norms $\|b\|_{BMO_r}$ and $\|b\|_{BMO}$ are equivalent. The equivalence of norms for $r > 1$ is well-known and follows from John-Nirenberg's lemma (see [8]), while the equivalence for $0 < r < 1$ has been proved by Hanks in [6]. (See also [12], page 179.)

For $r = 2$, it follows from the orthogonality of Haar system that

$$\|b\|_{BMO_2^d} = \left(\sup_{I \in \mathcal{D}} \frac{1}{|I|} \sum_{J \subseteq I} |\widehat{b}(J)|^2 \right)^{1/2}.$$

3. MULTILINEAR DYADIC PARAPRODUCTS AND HAAR MULTIPLIERS

We first recall the definitions of multilinear paraproduct operators and Haar multipliers introduced in [9].

For $m \geq 2$ and $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \{0, 1\}^m$, the *paraproduct operator* $P^{\vec{\alpha}}$ is defined by

$$P^{\vec{\alpha}}(f_1, f_2, \dots, f_m) = \sum_{I \in \mathcal{D}} \left(\prod_{j=1}^m f_j(I, \alpha_j) \right) h_I^{\sigma(\vec{\alpha})}$$

where $f_i(I, 0) = \langle f_i, h_I \rangle$, $f_i(I, 1) = \langle f_i \rangle_I$ and $\sigma(\vec{\alpha}) = \#\{i : \alpha_i = 0\}$.

Observe that if $\vec{\beta} = (\beta_1, \beta_2, \dots, \beta_m)$ is some permutation of $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m)$ and (g_1, g_2, \dots, g_m) is the corresponding permutation of (f_1, f_2, \dots, f_m) , then

$$P^{\vec{\alpha}}(f_1, f_2, \dots, f_m) = P^{\vec{\beta}}(g_1, g_2, \dots, g_m).$$

For a given function b , the paraproduct operator $\pi_b^{\vec{\alpha}}$ is defined by

$$\pi_b^{\vec{\alpha}}(f_1, f_2, \dots, f_m) = P^{(0, \vec{\alpha})}(b, f_1, f_2, \dots, f_m) = \sum_{I \in \mathcal{D}} \langle b, h_I \rangle \prod_{j=1}^m f_j(I, \alpha_j) h_I^{1+\sigma(\vec{\alpha})}$$

where $(0, \vec{\alpha}) = (0, \alpha_1, \dots, \alpha_m) \in \{0, 1\}^{m+1}$.

Note that

$$\pi_b^1(f) = P^{(0,1)}(b, f) = \sum_{I \in \mathcal{D}} b(I, 0) f(I, 1) h_I = \sum_{I \in \mathcal{D}} \langle b, h_I \rangle \langle f \rangle_I h_I = \pi_b(f).$$

Given a symbol sequence $\epsilon = \{\epsilon_I\}_{I \in \mathcal{D}}$, the *m-linear Haar multiplier* $T_\epsilon^{\vec{\alpha}}$ is defined by

$$T_\epsilon^{\vec{\alpha}}(f_1, f_2, \dots, f_m) := \sum_{I \in \mathcal{D}} \epsilon_I \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})}.$$

Let $1 \leq i \leq m$. For a given function g , we define

$$M_g^i(f_1, \dots, f_m) := (f_1, \dots, g f_i, \dots, f_m).$$

Note that if T is multilinear, so is $T(M_g^i)$, and for $g = 1$, $T(M_g^i) = T$.

The following property of the multilinear dyadic operators will be very useful for our purpose.

Lemma 3.1. *Let $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \{0, 1\}^m$, and let T be any of the m -linear operators $P^{\vec{\alpha}}$, $\pi_b^{\vec{\alpha}}$ or $T_\epsilon^{\vec{\alpha}}$. Then for a given function g and $J \in \mathcal{D}$, the function*

$$T(M_g^i(f_1, f_2, \dots, f_m)) - T(M_g^i(f_1 \mathbf{1}_J, f_2 \mathbf{1}_J, \dots, f_m \mathbf{1}_J))$$

is constant on J . In particular,

$$T(f_1, f_2, \dots, f_m) - T(f_1 \mathbf{1}_J, f_2 \mathbf{1}_J, \dots, f_m \mathbf{1}_J)$$

is constant on J .

Proof. Fix $J \in \mathcal{D}$. Let $f_i \mathbf{1}_J = f_i^0$ and $f_i - f_i \mathbf{1}_J = f_i^\infty$. Since $T(M_g^i)$ is multilinear,

$$\begin{aligned} T(M_g^i(f_1, f_2, \dots, f_m)) &= T(M_g^i(f_1^0 + f_1^\infty, f_2^0 + f_2^\infty, \dots, f_m^0 + f_m^\infty)) \\ &= T(M_g^i(f_1^0, f_2^0, \dots, f_m^0)) + \sum_{\substack{\vec{\beta} \in \{0, \infty\}^m \\ \vec{\beta} \neq \vec{0}}} T(M_g^i(f_1^{\beta_1}, f_2^{\beta_2}, \dots, f_m^{\beta_m})), \end{aligned}$$

where $\vec{\beta} = (\beta_1, \dots, \beta_m)$.

Observe that if $I \subseteq J$, $\widehat{f_j^\infty}(I) = \widehat{gf_j^\infty}(I) = \langle f_j^\infty \rangle_I = \langle gf_j^\infty \rangle_I = 0$, since each of the functions f_j^∞, gf_j^∞ is identically 0 on J . So for $\vec{\beta} \neq \vec{0}$,

$$T(M_g^i(f_1^{\beta_1}, f_2^{\beta_2}, \dots, f_m^{\beta_m})) = \sum_{I \in \mathcal{D}} \delta_J^T \prod_{j=1}^m F_j^{\beta_j}(I, \alpha_j) h_I^{\sigma(\vec{\alpha}, T)} = \sum_{I: I \not\subseteq J} \delta_J^T \prod_{j=1}^m F_j^{\beta_j}(I, \alpha_j) h_I^{\sigma(\vec{\alpha}, T)},$$

where

$$\delta_J^T = \begin{cases} 1, & \text{if } T = P^{\vec{\alpha}} \\ \widehat{b}(J), & \text{if } T = \pi_b^{\vec{\alpha}}, \\ \epsilon_J & \text{if } T = T_\epsilon^{\vec{\alpha}} \end{cases},$$

$$F_j^{\beta_j} = \begin{cases} f_j^{\beta_j}, & \text{if } j \neq i \\ gf_j^{\beta_j}, & \text{if } j = i \end{cases},$$

and

$$\sigma(\vec{\alpha}, T) = \begin{cases} \sigma(\vec{\alpha}), & \text{if } T = P^{\vec{\alpha}} \text{ or } T_\epsilon^{\vec{\alpha}} \\ \sigma(\vec{\alpha}) + 1, & \text{if } T = \pi_b^{\vec{\alpha}} \end{cases}.$$

Since each h_I with $I \not\subseteq J$ is constant on J , so is $T(M_g^i(f_1^{\beta_1}, f_2^{\beta_2}, \dots, f_m^{\beta_m}))$ for $\vec{\beta} \neq \vec{0}$.

Consequently, $\sum_{\substack{\vec{\beta} \in \{0, \infty\}^m \\ \vec{\beta} \neq \vec{0}}} T(M_g^i(f_1^{\beta_1}, f_2^{\beta_2}, \dots, f_m^{\beta_m}))$ is constant on J , say C_J . Then for every

$x \in J$,

$$T(M_g^i(f_1, f_2, \dots, f_m))(x) - T(M_g^i(T)(f_1 \mathbf{1}_J, f_2 \mathbf{1}_J, \dots, f_m \mathbf{1}_J))(x) = c_J.$$

Taking $g = 1$, we see that $T(f_1, f_2, \dots, f_m) - T(f_1 \mathbf{1}_J, f_2 \mathbf{1}_J, \dots, f_m \mathbf{1}_J)$ is constant on J . \square

Lemma 3.2. *Let $b \in BMO^d$, and $\epsilon = (\epsilon_I)_{I \in \mathcal{D}}$ be bounded. Let $T \in \{P^{\vec{\alpha}}, T_\epsilon^{\vec{\alpha}}\}$ with $\vec{\alpha} \in U_m$, or $T = \pi_b^{\vec{\alpha}}$ with $\vec{\alpha} \in \{0, 1\}^m$. Then for $0 < \delta < \frac{1}{m}$, and $\vec{f} = (f_1, f_2, \dots, f_m) \in L^{p_1} \times L^{p_2} \times \dots \times L^{p_m}$ with $1 \leq p_i < \infty$, we have*

$$M_\delta^\#(T(\vec{f}))(x) \lesssim \mathcal{M}(\vec{f})(x).$$

Proof. Fix a point x . We will show that for every dyadic interval I containing x , there exists a constant c_I such that

$$\left(\frac{1}{|I|} \int_I \left| |T(\vec{f})(y)|^\delta - |c_I|^\delta \right| dy \right)^{1/\delta} \lesssim \mathcal{M}(\vec{f})(x),$$

from which the assertion follows. In fact, since $\left| \left| T(\vec{f})(y) \right|^\delta - |c_I|^\delta \right| \leq \left| T(\vec{f})(y) - c_I \right|^\delta$ for $0 < \delta < 1$, it suffices to show that

$$\left(\frac{1}{|I|} \int_I \left| T(\vec{f})(y) - c_I \right|^\delta \right)^{1/\delta} \lesssim \mathcal{M}(\vec{f})(x).$$

Fix a dyadic interval I that contains x , and let $f_i^0 = f \mathbf{1}_I$, $f_i^\infty = f_i - f_i^0$.

Writing $\vec{f}^0 = (f_1^0, \dots, f_m^0)$, Lemma 3.1 says that $T(\vec{f})(y) - T(\vec{f}^0)(y)$ is constant for all y in I , say c_I . We then have $T(\vec{f})(y) - c_I = T(\vec{f}^0)(y)$ for all $y \in I$. So,

$$\left(\frac{1}{|I|} \int_I \left| T(\vec{f})(y) - c_I \right|^\delta \right)^{1/\delta} = \left(\frac{1}{|I|} \int_I \left| T(\vec{f}^0)(y) \right|^\delta \right)^{1/\delta}.$$

We can estimate this using the following form of Kolmogorov inequality:

If $0 < p < q < \infty$, then for any measurable function f , there exists a constant $C = C(p, q)$ such that

$$(3.1) \quad \|f\|_{L^p(I, \frac{dy}{|I|})} \leq C \|f\|_{L^{q,\infty}(I, \frac{dy}{|I|})}.$$

For $p = \delta, q = 1/m$ and $f = T(\vec{f}^0)$, (3.1) becomes

$$\left(\frac{1}{|I|} \int_I \left| T(\vec{f}^0)(y) \right|^\delta dy \right)^{1/\delta} \leq C \left\| T(\vec{f}^0)(y) \right\|_{L^{1/m,\infty}(I, \frac{dy}{|I|})}.$$

Now,

$$\begin{aligned} \left\| T(\vec{f}^0)(y) \right\|_{L^{1/m,\infty}(I, \frac{dy}{|I|})} &= \sup_{t>0} t \left(\frac{1}{|I|} \left| \left\{ y \in I : \left| T(\vec{f}^0)(y) \right| > t \right\} \right| \right)^m \\ &\leq \sup_{t>0} \frac{t}{|I|^m} \left| \left\{ y : \frac{1}{|I|^m} \left| T(\vec{f}^0)(y) \right| > \frac{t}{|I|^m} \right\} \right|^m \\ &= \sup_{t>0} \frac{t}{|I|^m} \left| \left\{ y : \left| T \left(\frac{f_1^0}{|I|}, \dots, \frac{f_m^0}{|I|} \right) (y) \right| > \frac{t}{|I|^m} \right\} \right|^m \\ &= \left\| T \left(\frac{f_1^0}{|I|}, \dots, \frac{f_m^0}{|I|} \right) (y) \right\|_{L^{1/m,\infty}}. \end{aligned}$$

Since $\frac{f_i^0}{|I|} \in L^1$ for all $1 \leq i \leq m$, it follows from the boundedness of $T : L^1 \times \dots \times L^1 \rightarrow L^{1/m, \infty}$ that

$$\begin{aligned} \left\| T \left(\frac{f_1^0}{|I|}, \dots, \frac{f_m^0}{|I|} \right) (y) \right\|_{L^{1/m, \infty}} &\lesssim \prod_{i=1}^m \left\| \frac{f_i^0}{|I|} \right\|_{L^1} \\ &= \prod_{i=1}^m \int \frac{|f_i^0|}{|I|} \\ &= \prod_{i=1}^m \frac{1}{|I|} \int_I |f_i| \\ &\leq \mathcal{M}(\vec{f})(x). \end{aligned}$$

This completes the proof. \square

The following lemma gives us the finiteness condition needed to apply Fefferman-Stein inequalities 2.1 and 2.2 for the multilinear dyadic operators.

Lemma 3.3. *Let $w \in A_\infty^d$ and $\vec{f} = (f_1, \dots, f_m)$ where each f_i is bounded and has compact support. If $\left\| \mathcal{M}(\vec{f}) \right\|_{L^p(w)} < \infty$ for some $p > 0$, then there exists a $\delta \in (0, 1/m)$ such that $\left\| M_\delta(T(\vec{f})) \right\|_{L^p(w)} < \infty$. Similarly, if $\left\| \mathcal{M}(\vec{f}) \right\|_{L^{p, \infty}(w)} < \infty$ for some $p > 0$, then there exists a $\delta \in (0, 1/m)$ such that $\left\| M_\delta(T(\vec{f})) \right\|_{L^{p, \infty}(w)} < \infty$.*

Proof. We prove the first assertion, the second one follows from similar arguments.

Since $w \in A_\infty^d$, it is in $A_{p_0}^d$ for some $p_0 > \max(1, pm)$. Then for any δ with $0 < \delta < p/p_0 < 1/m$, we have

$$\begin{aligned} \left\| M_\delta(T(\vec{f})) \right\|_{L^p(w)} &\leq \left\| M_{p/p_0}(T(\vec{f})) \right\|_{L^p(w)} \\ &= \left[\int_{\mathbb{R}} \left\{ \left(\sup_{I \ni x} \frac{1}{|I|} \int_I |T(\vec{f})|^{p/p_0} dt \right)^{p_0/p} \right\}^p dw(x) \right]^{1/p} \\ &= \left[\int_{\mathbb{R}} M(T(\vec{f})^{p/p_0})^{p_0} dw \right]^{\frac{1}{p_0} \times \frac{p_0}{p}} \\ &= \left\| M(T(\vec{f})^{p/p_0}) \right\|_{L^{p_0}(w)}^{p_0/p}, \end{aligned}$$

The boundedness of $M : L^{p_0}(w) \rightarrow L^{p_0}(w)$ for $w \in A_{p_0}^d$ gives

$$\left\| M(T(\vec{f})^{p/p_0}) \right\|_{L^{p_0}(w)} \lesssim \left\| T(\vec{f})^{p/p_0} \right\|_{L^{p_0}(w)}.$$

Consequently,

$$\begin{aligned}
 \|M_\delta(T(\vec{f}))\|_{L^p(w)} &\lesssim \|T(\vec{f})^{p/p_0}\|_{L^{p_0}(w)}^{p_0/p} \\
 &= \left(\int_{\mathbb{R}} |T(\vec{f})^{p/p_0}|^{p_0} dw \right)^{\frac{1}{p_0} \times \frac{p_0}{p}} \\
 &= \left(\int_{\mathbb{R}} |T(\vec{f})|^p dw \right)^{1/p} \\
 &= \|T(\vec{f})\|_{L^p(w)},
 \end{aligned}$$

So, it suffices to prove that $\|T(\vec{f})\|_{L^p(w)} < \infty$.

Since each f_i has compact support, there exist dyadic intervals $S' = [0, 2^{-k})$ and $S'' = [-2^{-k}, 0)$ such that the support of every f_i is contained in $S = S' \cup S''$.

To prove the assertion, it suffices to show that

$$\|T(\vec{f})\|_{L^p(S,w)} < \infty \quad \text{and} \quad \|T(\vec{f})\|_{L^p(\mathbb{R} \setminus S,w)} < \infty.$$

Since $w \in A_\infty^d$, $w^{1+\gamma} \in L_{loc}^1$ for sufficiently small γ , (see [11] or [5]). In particular, $w \in L^q(S)$ for $q := 1 + \gamma$. We can choose γ small enough so that $w \in L^q(S)$ and $q'p > \frac{1}{m}$. Then by Hölder's inequality, we have

$$\begin{aligned}
 \|T(\vec{f})\|_{L^p(S,w)} &= \left(\int_S |T(\vec{f})|^p w dx \right)^{1/p} \\
 &\leq \left(\left(\int_S |T(\vec{f})|^{pq'} dx \right)^{1/q'} \left(\int_S w^q dx \right)^{1/q} \right)^{1/p} \\
 &< \infty.
 \end{aligned}$$

Here, the finiteness of $\int_S |T(\vec{f})|^{pq'} dx$ follows from the boundedness of $T : L^{mpq'} \times \dots \times L^{mpq'} \rightarrow L^{pq'}$, and the fact that each f_i (being bounded with compact support) is in $L^{mpq'}$. We refer to [9] for the unweighted theory of multilinear dyadic operators.

To prove $\|T(\vec{f})\|_{L^p(\mathbb{R} \setminus S,w)} < \infty$, it suffices to show that

$$|T(\vec{f})(x)| \leq C\mathcal{M}(\vec{f})(x) \quad \text{for every } x \in \mathbb{R} \setminus S.$$

We prove this for $T = \pi_b^{\vec{\alpha}}$. Proofs for $P^{\vec{\alpha}}$ and $T_\epsilon^{\vec{\alpha}}$ follow similarly.

Fix $x \in \mathbb{R} \setminus S$. Let I_x be the smallest dyadic interval that contains x and one of the intervals S' and S'' .

For definiteness, assume $x > 0$. In this case I_x is the smallest dyadic interval containing x and S' . Note that if $x \notin I$, $h_I(x) = 0$ and, if $x \in I$ with $I \cap S' = \emptyset$, $f_j(I, \alpha_j) = 0$ for each j .

So,

$$\begin{aligned}
\left| \pi_b^{\vec{\alpha}}(\vec{f})(x) \right| &= \left| \sum_{I \in \mathcal{D}} \widehat{b}(I) \prod_{j=1}^m f_j(I, \alpha_j) h_I^{1+\sigma(\vec{\alpha})}(x) \right| \\
&= \left| \sum_{I \supseteq I_x} \widehat{b}(I) \prod_{j=1}^m f_j(I, \alpha_j) h_I^{1+\sigma(\vec{\alpha})}(x) \right| \\
&\leq \sum_{I \supseteq I_x} \frac{|\widehat{b}(I)|}{\sqrt{|I|}} \left(\prod_{j:\alpha_j=0} \frac{|\widehat{f}_j(I)|}{\sqrt{|I|}} \right) \left(\prod_{j:\alpha_j=1} |\langle f_j \rangle_I| \right) \mathbf{1}_I(x) \\
&\leq \|b\|_{BMO^d} \sum_{I \supseteq I_x} \left(\prod_{j:\alpha_j=0} \frac{|\widehat{f}_j(I)|}{\sqrt{|I|}} \right) \left(\prod_{j:\alpha_j=1} |\langle f_j \rangle_I| \right),
\end{aligned}$$

where the last inequality follows from the fact that for $b \in BMO^d$,

$$\frac{|\widehat{b}(I)|}{\sqrt{|I|}} \leq \left(\frac{1}{|I|} \sum_{J \subseteq I} |\widehat{b}(J)|^2 \right)^{1/2} \leq \|b\|_{BMO^d}.$$

Note that $\frac{|\widehat{f}_j(I)|}{\sqrt{|I|}} = \frac{1}{\sqrt{|I|}} \left| \int f_j h_I \right| \leq \frac{1}{\sqrt{|I|}} \int |f_j| \frac{\mathbf{1}_I}{\sqrt{|I|}} = \frac{1}{|I|} \int_I |f_j| = \langle |f_j| \rangle_I$, and since f_j is 0 on $\mathbb{R} \setminus S$, we have $\langle |f_j| \rangle_{I^1} = \frac{\langle |f_j| \rangle_I}{2}$ whenever I^1 is the parent of I with $I_x \subseteq I$. So, we have

$$\begin{aligned}
\left| \pi_b^{\vec{\alpha}}(\vec{f})(x) \right| &\leq \|b\|_{BMO^d} \sum_{I \supseteq I_x} \prod_{j=1}^m \langle |f_j| \rangle_I \\
&= \|b\|_{BMO^d} \left(\prod_{j=1}^m \langle |f_j| \rangle_{I_x} + \frac{1}{2^m} \prod_{j=1}^m \langle |f_j| \rangle_{I_x} + \frac{1}{2^{2m}} \prod_{j=1}^m \langle |f_j| \rangle_{I_x} + \cdots \right) \\
&= \frac{2^m}{(2^m - 1)} \|b\|_{BMO^d} \prod_{j=1}^m \langle |f_j| \rangle_{I_x} \\
&\leq \frac{2^m}{(2^m - 1)} \|b\|_{BMO^d} \mathcal{M}(\vec{f})(x).
\end{aligned}$$

The same proof works for $x < 0$ too. This completes the proof. \square

Theorem 3.4. Let $b \in BMO^d$, and $\epsilon = (\epsilon_I)_{I \in \mathcal{D}}$ be bounded. Let $T \in \{P^{\vec{\alpha}}, T_{\epsilon}^{\vec{\alpha}}\}$ with $\vec{\alpha} \in U_m$, or $T = \pi_b^{\vec{\alpha}}$ with $\vec{\alpha} \in \{0, 1\}^m$. Then for $w \in A_{\infty}^d$ and $p > 0$,

$$\|T(\vec{f})\|_{L^p(w)} \lesssim \|\mathcal{M}(\vec{f})\|_{L^p(w)}$$

and

$$\|T(\vec{f})\|_{L^{p,\infty}(w)} \lesssim \|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(w)}$$

for all m -tuples $\vec{f} = (f_1, \dots, f_m)$ of bounded functions with compact support.

Proof. To prove the first inequality, assume that $\|\mathcal{M}(\vec{f})\|_{L^p(w)} < \infty$, otherwise there is nothing to prove. Then by Lemma 3.3, there exists a $\delta \in (0, 1/m)$ such that $\left\|M_\delta \left(T(\vec{f})\right)\right\|_{L^p(w)} < \infty$. For such δ , we have

$$\left\|T(\vec{f})\right\|_{L^p(w)} \leq \left\|M_\delta \left(T(\vec{f})\right)\right\|_{L^p(w)} \leq C \left\|M_\delta^\# \left(T(\vec{f})\right)\right\|_{L^p(w)} \leq C \left\|\mathcal{M}(\vec{f})\right\|_{L^p(w)},$$

where the first and last inequalities follow from pointwise control and the second inequality is the Fefferman-Stein's inequality (2.1).

Proof of the second inequality follows similarly, by applying Lemma 3.3 and using the Fefferman-Stein's inequality (2.2) for weak-type estimates. \square

Theorem 3.5. *Let $b \in BMO^d$, and $\epsilon = (\epsilon_I)_{I \in \mathcal{D}}$ be bounded. Suppose $T \in \{P^\alpha, T_\epsilon^\alpha\}$ with $\vec{\alpha} \in U_m$, or $T = \pi_b^\alpha$ with $\vec{\alpha} \in \{0, 1\}^m$. Let $\vec{w} = (w_1, \dots, w_m) \in A_P^d$ for $\vec{P} = (p_1, \dots, p_m)$ with $\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p}$.*

(a) *If $1 < p_1, \dots, p_m < \infty$, then*

$$(3.2) \quad \|T(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

(b) *If $1 \leq p_1, \dots, p_m < \infty$, then*

$$(3.3) \quad \|T(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{w}})} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

Proof. Since the simple functions in $L^p(w)$ are dense in $L^p(w)$ for any weight w (see [1]), it suffices to prove the estimates for $\vec{f} = (f_1, f_2, \dots, f_m)$ with $f_i \in L^{p_i}(w_i)$ simple. Note that $\vec{w} = (w_1, \dots, w_m) \in A_P^d$ implies that $\nu_{\vec{w}} \in A_\infty^d$. So, by Theorem 3.4 and the boundedness properties of the multilinear maximal function \mathcal{M} , we have

$$\|T(\vec{f})\|_{L^p(\nu_{\vec{w}})} \lesssim \|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \lesssim \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)},$$

and

$$\|T(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{w}})} \lesssim \|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{w}})} \lesssim \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

\square

4. COMMUTATORS OF MULTILINEAR HAAR MULTIPLIERS

Definition 4.1. *Let $\vec{\alpha} \in U_m$ and $\epsilon = (\epsilon_I)_{I \in \mathcal{D}}$ be bounded. Given a locally integrable function b , we define the commutator $[b, T_\epsilon^\alpha]_i, 1 \leq i \leq m$, by*

$$[b, T_\epsilon^\alpha]_i(f_1, f_2, \dots, f_m)(x) := b(x)T_\epsilon^\alpha(f_1, f_2, \dots, f_m)(x) - T_\epsilon^\alpha(f_1, \dots, bf_i, \dots, f_m)(x). \\ \text{i.e. } [b, T_\epsilon^\alpha]_i = M_b \circ T_\epsilon^\alpha - T_\epsilon^\alpha \circ M_b^i.$$

Theorem 4.1. *Let $\vec{\alpha} \in U_m$ and $\epsilon = (\epsilon_I)_{I \in \mathcal{D}}$ be bounded. Let $\delta \in (0, 1/m)$ and $\gamma > \delta$. Then for any $r > 1$,*

$$(4.1) \quad M_\delta^\# \left([b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f}) \right) (x) \lesssim \|b\|_{BMO^d} \left(\mathcal{M}_r(\vec{f})(x) + M_\gamma \left(T_\epsilon^{\vec{\alpha}}(\vec{f}) \right) (x) \right)$$

for all m -tuples $\vec{f} = (f_1, f_2, \dots, f_m)$ of bounded measurable functions with compact support.

Proof. Fix $x \in \mathbb{R}$. As in the proof of Lemma 3.2, it suffices to show that for every $I \in \mathcal{D}$ containing x , there exists a constant C_I such that

$$\left(\frac{1}{|I|} \int_I \left| [b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})(t) - C_I \right|^\delta dt \right)^{1/\delta} \lesssim \|b\|_{BMO^d} \left(\mathcal{M}_r(\vec{f})(x) + M_\gamma \left(T_\epsilon^{\vec{\alpha}}(\vec{f}) \right) (x) \right).$$

Fix $I \in \mathcal{D}$ containing x , and take $C_I = T_\epsilon^{\vec{\alpha}} \left(M_g^i(\vec{f}^0) \right) (t) - T_\epsilon^{\vec{\alpha}} \left(M_g^i(\vec{f}) \right) (t)$, where $g = b - \langle b \rangle_I$ and $\vec{f}^0 = (f_1^0, \dots, f_m^0)$ with $f_i^0 = f_i \mathbf{1}_I$. Lemma 3.1 shows that this is indeed a constant on I . Since $T_\epsilon^{\vec{\alpha}}$ is multilinear,

$$\begin{aligned} [b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})(t) &= b(t) T_\epsilon^{\vec{\alpha}}(\vec{f})(t) - T_\epsilon^{\vec{\alpha}}(f_1, \dots, b f_i, \dots, f_m)(t) \\ &= (b(t) - \langle b \rangle_I) T_\epsilon^{\vec{\alpha}}(\vec{f})(t) - T_\epsilon^{\vec{\alpha}}(f_1, \dots, (b - \langle b \rangle_I) f_i, \dots, f_m)(t) \\ &= (b(t) - \langle b \rangle_I) T_\epsilon^{\vec{\alpha}}(\vec{f})(t) - T_\epsilon^{\vec{\alpha}} \left(M_g^i(\vec{f}) \right) (t). \end{aligned}$$

So,

$$\begin{aligned} & \left(\frac{1}{|I|} \int_I \left| [b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})(t) - C_I \right|^\delta dt \right)^{1/\delta} \\ &= \left(\frac{1}{|I|} \int_I \left| (b(t) - \langle b \rangle_I) T_\epsilon^{\vec{\alpha}}(\vec{f})(t) - T_\epsilon^{\vec{\alpha}} \left(M_g^i(\vec{f}) \right) (t) - C_I \right|^\delta dt \right)^{1/\delta} \\ &= \left(\frac{1}{|I|} \int_I \left| (b(t) - \langle b \rangle_I) T_\epsilon^{\vec{\alpha}}(\vec{f})(t) - T_\epsilon^{\vec{\alpha}} \left(M_g^i(\vec{f}^0) \right) (t) \right|^\delta dt \right)^{1/\delta} \\ &\lesssim \left(\frac{1}{|I|} \int_I \left| (b(t) - \langle b \rangle_I) T_\epsilon^{\vec{\alpha}}(\vec{f})(t) \right|^\delta dt \right)^{1/\delta} + \left(\frac{1}{|I|} \int_I \left| T_\epsilon^{\vec{\alpha}} \left(M_g^i(\vec{f}^0) \right) (t) \right|^\delta dt \right)^{1/\delta}. \end{aligned}$$

Note that $\gamma/\delta > 1$. For any $q \in (1, \gamma/\delta)$, Hölder's inequality gives

$$\begin{aligned} & \left(\frac{1}{|I|} \int_I \left| (b(t) - \langle b \rangle_I) T_\epsilon^{\vec{\alpha}}(\vec{f})(t) \right|^\delta dt \right)^{1/\delta} \\ &\leq \left(\frac{1}{|I|} \int_I |b(t) - \langle b \rangle_I|^{\delta q'} dt \right)^{1/\delta q'} \left(\frac{1}{|I|} \int_I \left| T_\epsilon^{\vec{\alpha}}(\vec{f})(t) \right|^{\delta q} dt \right)^{1/\delta q} \\ &\lesssim \|b\|_{BMO^d} M_{\delta q} \left(T_\epsilon^{\vec{\alpha}}(\vec{f}) \right) (x) \\ &\leq \|b\|_{BMO^d} M_\gamma \left(T_\epsilon^{\vec{\alpha}}(\vec{f}) \right) (x). \end{aligned}$$

As in the proof of Lemma 3.2, we can apply Kolmogorov's inequality to obtain

$$\begin{aligned}
 & \left(\frac{1}{|I|} \int_I |T_\epsilon^\alpha(f_1^0, \dots, (b - \langle b \rangle_I) f_i^0, \dots, f_m^0)(t)|^\delta dt \right)^{1/\delta} \\
 & \leq \|T_\epsilon^\alpha(f_1^0, \dots, (b - \langle b \rangle_I) f_i^0, \dots, f_m^0)(t)\|_{L^{\frac{1}{\delta}, \infty}(I, \frac{dt}{|I|})} \\
 & \leq \frac{1}{|I|} \int_I |b(t) - \langle b \rangle_I| f_i^0(t) dt \prod_{j=1, j \neq i}^m \frac{1}{|I|} \int_I |f_j^0(t)| dt \\
 & \leq \left(\frac{1}{|I|} \int_I |b(t) - \langle b \rangle_I|^{r'} dt \right)^{1/r'} \left(\frac{1}{|I|} \int_I |f_i^0(t)|^r dt \right)^{1/r} \prod_{j=1, j \neq i}^m \left(\frac{1}{|I|} \int_I |f_j^0(t)|^r dt \right)^{1/r} \\
 & \lesssim \|b\|_{BMO^d} \prod_{j=1}^m \left(\frac{1}{|I|} \int_I |f_j(t)|^r dt \right)^{1/r} \\
 & \leq \|b\|_{BMO^d} \mathcal{M}_r(\vec{f})(x).
 \end{aligned}$$

We thus have

$$M_\delta^\# \left([b, T_\epsilon^\alpha]_i(\vec{f}) \right)(x) \lesssim \|b\|_{BMO^d} \left(\mathcal{M}_r(\vec{f})(x) + M_\gamma \left(T_\epsilon^\alpha(\vec{f}) \right)(x) \right).$$

□

Lemma 3.3 is also true for the commutators of the multilinear Haar multipliers with a bounded function b .

Lemma 4.2. *Let $w \in A_\infty^d$ and $\vec{f} = (f_1, \dots, f_m)$ where each f_i is bounded and has compact support. If $\left\| \mathcal{M}(\vec{f}) \right\|_{L^p(w)} < \infty$ for some $p > 0$, and b bounded, then there exists a $\delta \in (0, 1/m)$ such that $\left\| M_\delta \left([b, T_\epsilon^\alpha]_i(\vec{f}) \right) \right\|_{L^p(w)} < \infty$.*

Proof. Since each f_i has compact support, there exist dyadic intervals $S' = [0, 2^{-k})$ and $S'' = [-2^{-k}, 0)$ such that the support of every f_i is contained in $S = S' \cup S''$.

Following the arguments used in the proof of Lemma 3.3, we get

$$\left\| M_\delta \left([b, T_\epsilon^\alpha]_i(\vec{f}) \right) \right\|_{L^p(w)} \leq \left\| [b, T_\epsilon^\alpha]_i(\vec{f}) \right\|_{L^p(w)}.$$

So, it suffices to prove that

$$\left\| [b, T_\epsilon^\alpha]_i(\vec{f}) \right\|_{L^p(S, w)} < \infty \quad \text{and} \quad \left\| [b, T_\epsilon^\alpha]_i(\vec{f}) \right\|_{L^p(\mathbb{R} \setminus S, w)} < \infty.$$

Since $w \in A_\infty^d$, $w^{1+\gamma} \in L_{loc}^1$ for sufficiently small γ , (see [11] or [5]). In particular, $w \in L^q(S)$ for $q := 1 + \gamma$. We can choose γ small enough so that $w \in L^q(S)$ and $q'p > 1$. Then by

Hölder's inequality, we have

$$\begin{aligned} \left\| [b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f}) \right\|_{L^p(S, w)} &= \left(\int_S \left| [b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f}) \right|^p w dx \right)^{1/p} \\ &\leq \left(\left(\int_S \left| [b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f}) \right|^{pq'} dx \right)^{1/q'} \left(\int_S w^q dx \right)^{1/q} \right)^{1/p} \\ &< \infty. \end{aligned}$$

Here, $\int_S w^q dx < \infty$ because $w \in L_{loc}^q$, and the finiteness of $\int_S \left| [b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f}) \right|^{pq'} dx$ follows from boundedness of $[b, T_\epsilon^{\vec{\alpha}}]_i : L^{mpq'} \times \dots \times L^{mpq'} \rightarrow L^{pq'}$, and the fact that each f_i (being bounded with compact support) is in $L^{mpq'}$. For the unweighted theory of the commutators of multilinear Haar multipliers we refer to [9]. Note that to prove finiteness of $\left\| [b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f}) \right\|_{L^p(S, w)}$ we may assume that the BMO function b is in some L^p space with $1 < p < \infty$. Indeed, for all $x \in S$,

$$[b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})(x) = [b \mathbf{1}_S, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})(x),$$

for all $\vec{f} = (f_1, \dots, f_m)$ with f_i supported in S .

Now to prove $\left\| [b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f}) \right\|_{L^p(\mathbb{R} \setminus S, w)} < \infty$, it suffices to show that for every $x \in \mathbb{R} \setminus S$,

$$\left| [b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})(x) \right| \leq \mathcal{M}(\vec{f})(x).$$

Fix $x \in \mathbb{R} \setminus S$. For definiteness, assume that $x > 0$, and let I_x be the smallest dyadic interval that contains x and the interval S' . Note that if $x \notin I$, $h_I(x) = 0$ and, if $x \in I$ with $I \cap S' = \emptyset$, $f_j(I, \alpha_j) = 0$ for each j . So,

$$\begin{aligned} &\left| [b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})(x) \right| \\ &\leq |b(x)| \left| T_\epsilon^{\vec{\alpha}}(f_1, f_2, \dots, f_m)(x) \right| + \left| T_\epsilon^{\vec{\alpha}}(f_1, \dots, bf_i, \dots, f_m)(x) \right| \\ &= |b(x)| \left| \sum_{I \supseteq I_x} \epsilon_I \prod_{j=1}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})}(x) \right| + \left| \sum_{I \supseteq I_x} \epsilon_I (bf_i)(I, \alpha_i) \prod_{\substack{j=1 \\ j \neq i}}^m f_j(I, \alpha_j) h_I^{\sigma(\vec{\alpha})}(x) \right| \\ &\leq |b(x)| \sum_{I \supseteq I_x} |\epsilon_I| \left(\prod_{j: \alpha_j=0} \frac{|\widehat{f_j}(I)|}{\sqrt{|I|}} \right) \left(\prod_{j: \alpha_j=1} |\langle f_j \rangle_I| \right) \mathbf{1}_I(x) \\ &\quad + |b(x)| \sum_{I \supseteq I_x} |\epsilon_I| |(bf_i)(I, \alpha_i)| \left(\prod_{\substack{j: \alpha_j=0 \\ j \neq i}} \frac{|\widehat{f_j}(I)|}{\sqrt{|I|}} \right) \left(\prod_{\substack{j: \alpha_j=1 \\ j \neq i}} |\langle f_j \rangle_I| \right) \mathbf{1}_I(x) \end{aligned}$$

We have $\frac{|\widehat{f_j}(I)|}{\sqrt{|I|}} = \frac{1}{\sqrt{|I|}} \left| \int f_j h_I \right| \leq \frac{1}{\sqrt{|I|}} \int |f_j| \frac{1_I}{\sqrt{|I|}} = \frac{1}{|I|} \int_I |f_j| = \langle |f_j| \rangle_I$. Since f_j is 0 on $\mathbb{R} \setminus S$, $\langle |f_j| \rangle_{I^1} = \frac{\langle |f_j| \rangle_I}{2}$ whenever I^1 is the parent of I with $I_x \subseteq I$. Moreover, $|(bf_i)(I, \alpha_i)| \leq |b| \langle |f_i| \rangle_I$. So,

$$\begin{aligned}
 & \left| [b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f})(x) \right| \\
 & \leq 2 \left(\sup_{x \in \mathbb{R}} |b(x)| \right) \left(\sup_{I \in \mathcal{D}} |\epsilon_I| \right) \sum_{I \supseteq I_x} \left(\prod_{j: \alpha_j=0} \frac{|\widehat{f_j}(I)|}{\sqrt{|I|}} \right) \left(\prod_{j: \alpha_j=1} \langle |f_j| \rangle_I \right) \\
 & \leq 2 \left(\sup_{x \in \mathbb{R}} |b(x)| \right) \left(\sup_{I \in \mathcal{D}} |\epsilon_I| \right) \sum_{I \supseteq I_x} \prod_{j=1}^m \langle |f_j| \rangle_I \\
 & = 2 \left(\sup_{x \in \mathbb{R}} |b(x)| \right) \left(\sup_{I \in \mathcal{D}} |\epsilon_I| \right) \left(\prod_{j=1}^m \langle |f_j| \rangle_{I_x} + \frac{1}{2^m} \prod_{j=1}^m \langle |f_j| \rangle_{I_x} + \frac{1}{2^{2m}} \prod_{j=1}^m \langle |f_j| \rangle_{I_x} + \cdots \right) \\
 & = 2 \left(\frac{2^m}{2^m - 1} \right) \left(\sup_{x \in \mathbb{R}} |b(x)| \right) \left(\sup_{I \in \mathcal{D}} |\epsilon_I| \right) \prod_{j=1}^m \langle |f_j| \rangle_{I_x} \\
 & \leq \frac{2^{m+1}}{(2^m - 1)} \left(\sup_{x \in \mathbb{R}} |b(x)| \right) \left(\sup_{I \in \mathcal{D}} |\epsilon_I| \right) \mathcal{M}(\vec{f})(x).
 \end{aligned}$$

The same proof works for $x < 0$ with I_x the smallest dyadic interval that contains both x and the interval S'' .

□

Theorem 4.3. *Let $\vec{\alpha} \in U_m$ and $\epsilon = (\epsilon_I)_{I \in \mathcal{D}}$ be bounded. Suppose $b \in BMO^d$ and $\vec{w} = (w_1, \dots, w_m) \in A_{\vec{P}}^d$ for $\vec{P} = (p_1, \dots, p_m)$ with $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$ and $1 < p_1, \dots, p_m < \infty$. Then there exists a constant C such that*

$$(4.2) \quad \left\| [b, T_\epsilon^{\vec{\alpha}}]_i(\vec{f}) \right\|_{L^p(\nu_{\vec{w}})} \leq C \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

Proof. First assume that b is bounded.

Since the simple functions in $L^p(\nu_{\vec{w}})$ are dense in $L^p(\nu_{\vec{w}})$, it suffices to prove (4.2) for $\vec{f} = (f_1, f_2, \dots, f_m)$ with $f_i \in L^{p_i}(w_i)$ simple. For all such \vec{f} , there exists, by Lemma 4.2, a

$\delta \in (0, 1/m)$ such that $\left\| M_\delta \left([b, T_\epsilon^\alpha]_i(\vec{f}) \right) \right\|_{L^p(w)} < \infty$. So, for any $r > 1$ and $\gamma > \delta$ we have

$$\begin{aligned} \left\| [b, T_\epsilon^\alpha]_i(\vec{f}) \right\|_{L^p(\nu_{\vec{w}})} &\leq \left\| M_\delta [b, T_\epsilon^\alpha]_i(\vec{f}) \right\|_{L^p(\nu_{\vec{w}})} \\ &\lesssim \left\| M_\delta^\# [b, T_\epsilon^\alpha]_i(\vec{f}) \right\|_{L^p(\nu_{\vec{w}})} \\ &\lesssim \|b\|_{BMO^d} \left(\left\| \mathcal{M}_r(\vec{f}) \right\|_{L^p(\nu_{\vec{w}})} + \left\| M_\gamma \left(T_\epsilon^\alpha(\vec{f}) \right) \right\|_{L^p(\nu_{\vec{w}})} \right), \end{aligned}$$

where the first inequality follows from the pointwise control, the second one is the Fefferman-Stein inequality (2.1) and the last inequality follows from Theorem 4.1.

Now we can choose $\gamma \in (\delta, 1/m)$ such that $\left\| M_\gamma \left(T_\epsilon^\alpha(\vec{f}) \right) \right\|_{L^p(\nu_{\vec{w}})} < \infty$. In fact, looking at the proofs of Lemmas 3.3 and 4.2, any $\gamma \in (\delta, p/p_0)$ would work. For such γ , we have

$$\begin{aligned} \left\| M_\gamma \left(T_\epsilon^\alpha(\vec{f}) \right) \right\|_{L^p(\nu_{\vec{w}})} &\lesssim \left\| M_\gamma^\# \left(T_\epsilon^\alpha(\vec{f}) \right) \right\|_{L^p(\nu_{\vec{w}})} \\ &\leq \left\| \mathcal{M}(\vec{f}) \right\|_{L^p(\nu_{\vec{w}})} \\ &\leq \left\| \mathcal{M}_r(\vec{f}) \right\|_{L^p(\nu_{\vec{w}})} \end{aligned}$$

We thus have

$$\left\| [b, T_\epsilon^\alpha]_i(\vec{f}) \right\|_{L^p(\nu_{\vec{w}})} \lesssim \|b\|_{BMO^d} \left\| \mathcal{M}_r(\vec{f}) \right\|_{L^p(\nu_{\vec{w}})}$$

for all $r > 1$.

Finally, we can choose $r > 1$ such that the inequality (2.5) holds, i.e.

$$\left\| \mathcal{M}_r(\vec{f}) \right\|_{L^p(\nu_{\vec{w}})} \lesssim \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

This completes the proof when b is bounded.

Now following [10], we use a limiting argument to prove the theorem for general $b \in BMO^d$. Let $\{b_j\}$ be the sequence of functions defined by

$$b_j(x) = \begin{cases} j, & \text{if } b(x) > j, \\ b(x), & \text{if } |b(x)| \leq j, \\ -j & \text{if } b(x) < -j. \end{cases}$$

Clearly, $b_j \rightarrow b$ pointwise, and we have $\|b_j\|_{BMO^d} \leq c\|b\|_{BMO^d}$ for all j . In fact, $c = 9/4$ works (see [5], page 129).

For any $q \in (1, \infty)$,

$$T_\epsilon^\alpha(f_1, \dots, b_j f_i, \dots, f_m) \rightarrow T_\epsilon^\alpha(f_1, \dots, b f_i, \dots, f_m) \quad \text{in } L^q \text{ as } j \rightarrow \infty$$

due to boundedness of $T_\epsilon^\alpha : L^{mq} \times \dots \times L^{mq} \rightarrow L^q$ and the fact that bounded functions f_1, \dots, f_m with compact support are all in L^{mq} . Note that since $b_j, b \in BMO^d$ and bounded

function f_i has compact support $b_j f_i \rightarrow b f_i$ in L^{mq} as $j \rightarrow \infty$. Then there exists a subsequence $\{b_{j_k}\}$ such that

$$T_\epsilon^\alpha(f_1, \dots, b_{j_k} f_i, \dots, f_m)(x) \rightarrow T_\epsilon^\alpha(f_1, \dots, b f_i, \dots, f_m)(x) \quad \text{for almost every } x.$$

For such x , we have

$$[b_{j_k}, T_\epsilon^\alpha]_i(\vec{f})(x) \rightarrow [b, T_\epsilon^\alpha]_i(\vec{f})(x).$$

Now,

$$\begin{aligned} \left\| [b, T_\epsilon^\alpha]_i(\vec{f}) \right\|_{L^p(\nu_{\vec{w}})} &= \left(\int_{\mathbb{R}} \left| [b, T_\epsilon^\alpha]_i(\vec{f})(x) \right|^p dx \right)^{1/p} \\ &\leq \liminf_{k \rightarrow \infty} \left(\int_{\mathbb{R}} \left| [b_{j_k}, T_\epsilon^\alpha]_i(\vec{f})(x) \right|^p dx \right)^{1/p} \\ &\leq C' \liminf_{k \rightarrow \infty} \|b_{j_k}\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)} \\ &\leq C \|b\|_{BMO^d} \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}, \end{aligned}$$

where we have used Fatou's lemma to obtain the first inequality, and the second inequality follows from the result already proved for bounded function b . \square

Some Remarks:

- (1) In [9] we have presented the unweighted theory of the multilinear commutators with some restrictions. In that paper, we required that $b \in L^q$ for some $q \in (1, \infty)$ and that $p > 1$. As we have seen, this restricted unweighted theory was sufficient to obtain the weighted theory presented in this article. Taking $w_i = 1$ for all $1 \leq i \leq m$, we see that the weighted theory implies the unweighted theory for all $b \in BMO^d$ and $1/m < p < \infty$.
- (2) With the results obtained in this article, it is easy to see that the end-point results obtained in [10] for the commutators of the multilinear Calderón-Zygmund operators also hold for the commutators of the multilinear Haar multipliers.

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